Comm. Alg. Sheet 2

1. Let F be a field and let Y be a set of polynomials in k variables over F. Prove that there exist finitely many polynomials $f_1, \ldots, f_m \in Y$ such that for $u_1, \ldots, u_k \in F^k$,

$$f(u_1, \ldots, u_k) = 0 \ \forall f \in Y \iff f_j(u_1, \ldots, u_k) = 0 \text{ for } j = 1, \ldots, m.$$

2. Let W be an algebraic set in F^k (F a field). Put $I = \mathcal{I}(W)$. Prove:
(i) $W = \mathcal{V}(I)$. (ii) $I = \mathrm{rad}(I)$.
(iii) Both F^k and $\varnothing$ are algebraic sets.
(iv) The union of two algebraic sets in F^k is an algebraic set.
(v) The intersection of any collection of algebraic sets in F^k is an algebraic set.

Remark. (iii) – (v) say that the set of all algebraic sets in F^k is the set of closed sets in a certain topology: this is called the *Zariski topology.*

3. An algebraic set W is *irreducible* if it is not the union of two proper algebraic subsets. (i) Prove that this holds if and only if $\mathcal{I}(W)$ is a prime ideal. (ii) Prove that every algebraic set is the union of finitely many irreducible algebraic subsets. (*Hint*: consider a counterexample W with $\mathcal{I}(W)$ as big as possible.) (iii) What's wrong with the following argument: If $I = \mathcal{I}(W)$ then $I = \mathrm{rad}(I) = P_1 \cap \ldots \cap P_m$ with the P_i minimal primes of I. Then $W = \bigcup V_i$ where $V_i = \mathcal{V}(P_i)$ is irreducible?

4. Let F be a field, $R = F[t_1, \ldots, t_k]$. For $\mathbf{u} = (u_1, \ldots, u_k) \in F^k$ define $e_{\mathbf{u}} : R \to F$ by $e_{\mathbf{u}}(f) = f(\mathbf{u}) = f(u_1, \ldots, u_k)$. Put

$$\mu(\mathbf{u}) = \sum_{i=1}^{k} (t_i - u_i)R.$$

(i) Prove that $R = \mu(\mathbf{u}) \oplus F$. Deduce (ii) $\mu(\mathbf{u}) \underset{\max}{\lhd} R$, (iii) $\mu(\mathbf{u}) = \ker e_{\mathbf{u}}$, (iv) $\mu(\mathbf{u}) = \mu(\mathbf{v}) \iff \mathbf{u} = \mathbf{v}$. (v) Prove: an ideal I of R is of the form $\mu(\mathbf{u})$ for some $\mathbf{u} \in F^k$ if and only if it has codimension one, i.e. $\dim_F(R/I) = 1$.

5. (i) Let F be a field, $R = F[t_1, \ldots, t_k]$. Let Y be a subset of R. Then $\mu(\mathcal{V}(Y))$ is a set of maximal ideals of R: identify this set. (Now you have transformed geometry into algebra!)
(ii) Suppose that F *not* algebraically closed; prove that not every maximal ideal of R is of the form $\mu(\mathbf{u})$.

6. Let F be an algebraically closed field and $f_1, \ldots, f_n$ polynomials in k variables over F. The system of simultaneous equations

$$\mathcal{F} : f_1(x_1, \ldots, x_k) = 0, \ldots, f_n(x_1, \ldots, x_k) = 0$$

is said to be *inconsistent* if there exist polynomials $g_1, \ldots, g_n$ such that $\sum_{i=1}^{n} f_i g_i = 1$.

(i) Prove that the system of equations $\mathcal{F}$ has a solution if and only if it is not inconsistent.

(ii) Suppose that the f_i have all coefficients in $\mathbb{Q}$, and that the system $\mathcal{F}$ has a solution in $\mathbb{C}^k$. Prove that it has a solution $(x_1, \ldots, x_k)$ with each x_i an algebraic number (i.e. algebraic over $\mathbb{Q}$).